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Can Additive Processes be Stationary?

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ABSTRACT
Three important types of processes are (a) those with independent increments (b) those with stationary & independent increments and (c) stationary processes. Brownian motion and Poisson processes belong to the second type. An interesting question here is; can processes belonging to types (a) or (b) be stationary? We prove that a stationary process cannot have independent increments and conversely.

KEYWORDS
Additive processes, Brownian motion, Lévy Processes, Poisson process, processes with independent increments, processes with stationary and independent increments, stationary processes.

1. Introduction

Processes with independent increments are also called additive processes and those with stationary & independent increments Lévy processes. We will use these terminologies in this note. Relationships among different types of processes are of interest, *viz.* whether a certain type of process is Markovian, stationary or martingale *etc.* While introducing classical types of stochastic processes, Karlin and Taylor (1975, p.30) sates: "Neither the Poisson process nor the Brownian motion is stationary. In fact no non-constant process with stationary independent increments is stationary". However, no proof of this is given in that book. In this note we briefly review the context to record some results discussing some relationships among the three types of processes mentioned in the abstract, and prove that a process cannot be additive and be stationary at the same time.

2. Relationships among the three types

Both Brownian motion and Poisson processes are Lévy processes but are not stationary. This is clear from the fact that their mean function is a function of t, and for a stationary process the mean function must be a constant, free of t, for all t.

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It is known (Karlin and Taylor, 1975, p.27) that if the increments of a Lévy process $\{X(t)\}$ have a finite mean, then the mean function $E\{X(t)\} = m_0 + m_1 t$, where $m_0 = E\{X(0)\}$ and $m_1 = E\{X(1)\} - m_0$. Of course, here it is implicitly assumed that $E\{X(1)\} \neq E\{X(0)\}$. Thus, a Lévy process $\{X(t)\}$ having a finite mean cannot be stationary. Now, what if $E\{X(t)\}$ is not finite? For example, Mittag-Leffler processes (Pillai, 1990) do not have a finite mean. More generally we have,

Theorem 2.1. *Lévy processes are not stationary.*

Proof. (Saz, 2019) Let $\{X(t), t \geq 0\}$ be a Lévy process. Since

$$X(t+h) \stackrel{d}{=} [X(t+h) - X(t)] + X(t)$$

it follows from the independence and stationarity of the increments that the respective characteristic functions satisfy

$$E[e^{iuX(t+h)}] = E[e^{iuX(h)}]\ E[e^{iuX(t)}], \forall\ u \in \mathbb{R}.$$

Now, if $\{X(t)\}$ is stationary, then $\{X(t)\}$ and $\{X(t+h)\}$ have the same distribution and so

$$E[e^{iuX(t)}] = E[e^{iuX(h)}]\ E[e^{iuX(t)}].$$

Since the increments of a Lévy process are infinitely divisible, its characteristic function cannot have real zeroes and hence $E[e^{iuX(h)}] = 1, \forall\ u$. By the uniqueness of characteristic functions we therefore get that $X(h) = 0\ a.s.$ However, this is possible only if $h = 0$, by the Lévy-Khintchin formula, and hence $\{X(t)\}$ and $\{X(t+h)\}$ can have the same distribution only in the trivial case $h = 0$. □

The question answered above came to this author's mind while trying to develop Laplace processes as a possible alternative to Brownian motion, during 1987-88, Satheesh (1990). We got the stronger result, theorem 2.3, below. We now ask; what if the increments of $\{X(t)\}$ are only independent and not stationary, that is, if $\{X(t)\}$ is additive? The following result in Feller (1966) is the crucial one in our argument.

Theorem 2.2. *(Feller, 1966, p.351) Let F be a probability distribution not concentrated at the origin, and let ξ be a bounded continuous solution to the convolution equation $\xi(x) = \int_{-\infty}^{\infty} \xi(x-y)F\{dy\}$, that is, $\xi = F * \xi$. Then $\xi(x)$ is a constant except if F is arithmetic. If F has span λ, then ξ is periodic with period λ.*

Note that in the 2^{nd} edition of this book (Feller, 1971), this theorem is not given as such, though discussed. We now have the following result.

Theorem 2.3. *An additive process cannot be stationary, and conversely.*

Proof. Suppose $\{X(t), t \geq 0\}$ is stationary, then

$$X(t) \stackrel{d}{=} X(s),\ \forall\ s, t. \tag{1}$$

If $\{X(t)\}$ is additive, then re-writing the left-hand-side of (1) for $s < t$,

$$X(s) + [X(t) - X(s)] \stackrel{d}{=} X(s). \tag{2}$$

In terms of the distribution functions ξ of $X(s)$ and F of $X(t) - X(s)$, (2) is, $\xi * F = \xi$. Now Theorem 2.2 implies that ξ cannot be a distribution function.

On the other hand, if we start with $\{X(t)\}$ being additive and then impose that $\{X(t)\}$ is stationary, we again arrive at

$$X(s) + X(t) - X(s) \stackrel{d}{=} X(t) \stackrel{d}{=} X(s), \forall\ s < t.$$

Once again, invoking Theorem 2.2 the proof is completed. □

Finally, we also observe the following interesting result in the context.

Theorem 2.4. *Stationary processes have stationary increments.*

Proof. (Kavi Rama Murthy, 2021) Suppose the process $\{X(t), t \geq 0\}$ is stationary and consider the increment $X(t+s) - X(t)$. Since $(X(t), X(t+s))$ has the same distribution as $(X(0), X(s))$, it follows that the increments $X(t+s) - X(t)$ and $X(s) - X(0)$ have the same distribution. □

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References

[1] Feller, W. (1966), *An Introduction to Probability Theory and Its Applications*, Vol.II, 1^{st} Edition, John Wiley & Sons, New York.

[2] Feller, W. (1971), *An Introduction to Probability Theory and Its Applications*, Vol.II, 2^{nd} Edition, John Wiley & Sons, New York.

[3] Karlin, S. and Taylor, H. M. (1975), *A First Course in Stochastic Processes*, 2^{nd} Edition, Academic Press, New York.

[4] Kavi Rama Murthy (2021), https://math.stackexchange.com/questions/4017261/stationary-processes-versus-processes-with-stationary-increments.

[5] Pillai, R. N. (1990), On Mittag-Leffler functions and related distributions, *Ann. Inst. Statist. Math.*, 42, 157-161.

[6] Satheesh, S (1990), Laplace process, *Current Science*, 59, 44.

[7] Saz (2019), https://math.stackexchange.com/questions/3315330/why-are-processes-with-stationary-independent-increments-nonstationary?